On convexity of Invariant Square mean root with respect to classical means

E. Dhanalakshmi[1], G. Narasimhan[2], K. M. Nagaraja[3*], R. Sampath Kumar[4]

[1]Department of Mathematics, SJC Institute of Technology, Chikkaballapur–562 101, India and affiliated to Visveswaraya Technological University, Belagavi–18, India.

[2,4]Department of Mathematics, RNS Institute of Technology Bengaluru-98, India, affiliated to Visveswaraya Technological University, Belagavi–18, India.

[3]Department of Mathematics, J S S Academy of Technical Education, Bengaluru-60, India and affiliated to Visveswaraya Technological University, Belagavi–18, India.

Abstract. In this paper, it is established that the square mean root is a hybrid mean, relationship between G – Complementary to square mean root and root mean square is obtained, moreover, some convexity properties with respect to classical means are discussed.

Key Word: Hybrid mean , Square mean root , Inverse square mean root , Root mean square , Properties of Convexity.

1. Introduction

The concept of mathematical means was introduced and studied by Greek mathematicians during the fourth century A.D., particularly within the Pythagorean School, where it was rooted in the study of proportions and their significance [1]. Over time, numerous researchers have contributed to and advanced this field, motivated by its wide-ranging applications in various branches of science and technology.

Relationships between series and important means [2], exploring the connections between Greek means and functional means [3], and introducing and studying the Gnan mean for both two and n variables [4]. They also investigated homogeneous functions and, as an application, derived several inequalities involving means [5].

Furthermore, they studied the Oscillatory mean and Oscillatory-type means in the context of Greek means. Their work led to the proposal of several new means, their generalizations, and numerous related inequality results ([6], [7], [8], [9], [10], [11], [12]). A substantial body of work on Schur-convexity and properties of means has been documented in ([13],[14],[15],[16],[17],[18],[19]) while studies focused on the refinement of mean inequalities can be found in ([20], [21], [22]).

In the literature, it is well known that for any two real numbers a and b the Arithmetic mean $A(a\,,b) = \frac{a+b}{2}$, Geometric mean $G(a\,,b) = \sqrt{ab}$, Harmonic mean $H(a\,,b) = \frac{2ab}{a+b}$ and Contra Harmonic mean $C(a\,,b) = \frac{a^2+b^2}{a+b}$ represent fundamental types of means frequently studied in mathematical analysis.

2. Definitions and Lemmas

Few basic definitions and lemmas are listed which are associated to this study.

Definition 2.1:[1] A Mean is defined as a function $M: R_+^n \to R_+$ which satisfies the following properties:

$$\min(a_1, a_2, a_{3,} \dots, a_n) \le \mathrm{M}(a_1, a_2, a_{3,} \dots, a_n) \le \max(a_1, a_2, a_{3,} \dots, a_n), \forall\, a_i \ge 0,\ 1 \le i \le n,$$

Definition 2.2:[1] For all real numbers a and b, the Power mean is defined as;

$$M_r(a\,,b) = \begin{cases} \left(\frac{a^r + b^2}{2}\right)^{\frac{1}{r}}, & r \neq 0 \\ \sqrt{ab}\ , & r = 0 \end{cases} \tag{1}$$

Remark: For $r = 1$, $M_1(a\,,b) = \frac{a\,+\,b}{2}$ is Arithmetic mean (AM).

For $r = 2$, $M_2(a\,,b) = \sqrt{\frac{a^2+b^2}{2}}$ is Root mean square (RMS).

Definition 2.3: [1] A mean N is called P – Complementary to M if it satisfies $P(M\,,N) = P$.

Suppose a given mean M has a unique G – Complementary mean N is denoted by

$N\ = \mathrm{M}^{(\mathrm{G})} = \frac{G^2}{M}$. The G – Complementary mean is called Inverse.

Lemma 2.4:[1] For $\phi(x) = x^2$ and $a = (a_0,\ a_1,\ a_2)$ is just the determinant of Vander Monde's matrix of the 2^{nd} order takes the form.

$$V(a; r = 2, k = 0) = \begin{vmatrix} 1 & a_0 & a_0^2 \\ 1 & a_1 & a_1^2 \\ 1 & a_2 & a_2^2 \end{vmatrix} = (a_1 - a_0)(a_2 - a_0)(a_2 - a_1).$$

Lemma 2.5:[1] Let $f(x)$ and $g(x)$ be two functions, then $f(x)$ is said to be convex with respect to $g(x)$ for $a \leq b \leq c$ if and only if

$$\begin{vmatrix} 1 & f(a) & g(a) \\ 1 & f(b) & g(b) \\ 1 & f(c) & g(c) \end{vmatrix} \geq 0 \text{ which is equivalent to } \begin{vmatrix} 1 & f(a) & g(a) \\ 0 & f(b) - f(a) & g(b) - g(a) \\ 0 & f(c) - f(a) & g(c) - g(a) \end{vmatrix} \geq 0.$$

Lemma 2.6:[1] Let $\Omega\ \subseteq\ R^n$ be symmetric with non−empty interior geometrically convex set and let $\varphi :\ \Omega\ \rightarrow R_+$ be continuous on Ω and differentiable in Ω^0. If φ is symmetric on Ω then φ is

$$i)\ (x_1 - x_2)\left(\frac{\partial\varphi}{\partial x_1} - \frac{\partial\varphi}{\partial x_2}\right) \geq 0\ (\leq 0)$$

$$ii)\ (ln\,x_1 - ln\,x_2)\left(x_1\frac{\partial\varphi}{\partial x_1} - x_2\frac{\partial\varphi}{\partial x_2}\right) \geq 0\ (\leq 0)$$

$$iii)\ (x_1 - x_2)\left({x_1}^2\frac{\partial\varphi}{\partial x_1} - {x_2}^2\frac{\partial\varphi}{\partial x_2}\right) \geq 0\ (\leq 0)$$

is a Schur convex (concave), Schur−geometrically convex (concave) and Schur−Harmonically convex (concave) function respectively.

3. Square Mean Root

This section provides the definition and properties of square mean root.

Definition 3.1: For any real numbers a and b the square mean root is defined as:

$$SMR(a\,,b) = \left(\frac{\sqrt{a}\ +\ \sqrt{b}}{2}\right)^2.$$

Remark 3.2: This is a particular case of power mean, $M_{\frac{1}{2}}(a\,,b) = \left(\frac{\sqrt{a}\ +\sqrt{b}}{2}\right)^2$.

This is also a hybrid mean, $A(A\,,G) = \frac{1}{2}(A+G) = \frac{1}{2}\left(\frac{a+b}{2} + \sqrt{ab}\right) = \left(\frac{\sqrt{a}\,+\sqrt{b}}{2}\right)^2$.

Definition 3.3: The inverse of square mean root is defined as ${}^{i}SMR(a\,,b) = \frac{4ab}{(\sqrt{a}\,+\sqrt{b})^2}$.

4. Methodology

The analytical methods are applied in this article to prove the theorems by the use of partial derivatives and Taylor's series expansions. To justify the convexity of the graphs are obtained by the software, Origin.

5. Results and Discussions

This section provides the important results pertaining to G−Complementary to square root mean.

Theorem 5.1: For $a < b$, verify that G−Complementary to square mean root is a mean.

Proof: Case (1): For $a < b$, Consider, ${}^{i}SMR(a\,,b) - a = \frac{4ab}{(\sqrt{a}\,+\sqrt{b}\,)^2} - a = \frac{a\,(\,3b - a - 2\sqrt{ab}\,)}{(\sqrt{a}\,+\sqrt{b}\,)^2} > 0$

Which gives, $a <\ {}^{i}SMR(a\,,b)$ and hence Min $(a\,,b) =\ {}^{i}SMR(a\,,b)$

Case (2): For $a <$ b, Consider, $b -\ {}^{i}SMR(a\,,b) = b - \frac{4ab}{(\sqrt{a}\,+\sqrt{b}\,)^2} = \frac{b\,(\,b - 2\sqrt{ab}\, - 3a\,)}{(\sqrt{a}\,+\sqrt{b}\,)^2} > 0$

Which gives, ${}^{i}SMR(a\,,b) <\ b$ and hence ${}^{i}SMR(a\,,b)\ <\ $ Max $(a\,,b)$.

Combining both the above cases, Min $(a\,,b) <\ {}^{i}SMR(a\,,b) <$ Max $(a\,,b)$.

Therefore, ${}^{i}SMR(a\,,b)$ is a Mean. Hence the proof of the theorem.

Theorem 5.2: For $a < b$, the following inequality holds:

$$H < {}^{i}SMR(a\,,b) < G < A < C < H_e < RSM.$$

Proof: This is to establish an inequality chain involving the G−Complementary to square mean root (${}^{i}SMR$).

$${}^{i}SMR(a\,,b) - A(a\,,b) = \frac{4ab}{(\sqrt{a}\,+\sqrt{b}\,)^2} - \left(\frac{a+b}{2}\right) = \frac{-a^2 - b^2 + 6ab - 2\sqrt{ab}\,(a+b)}{2(\sqrt{a}\,+\sqrt{b}\,)^2} < 0$$

$${}^{i}SMR(a\,,b) - G(a\,,b) = \frac{4ab}{(\sqrt{a}\,+\sqrt{b}\,)^2} - \sqrt{ab} = \frac{2ab - \sqrt{ab}\,(a+b)}{(\sqrt{a}\,+\sqrt{b}\,)^2} < 0$$

$${}^{i}SMR(a\,,b) - H(a\,,b) = \frac{4ab}{(\sqrt{a}\,+\sqrt{b}\,)^2} - \frac{2ab}{a+b} = \frac{2a^2b + 2b^2a - 4ab\sqrt{ab}}{(a+b)(\sqrt{a}\,+\sqrt{b}\,)^2} > 0$$

$${}^{i}SMR(a\,,b) - C(a\,,b) = \frac{4ab}{(\sqrt{a}\,+\sqrt{b}\,)^2} - \left(\frac{a^2+b^2}{a+b}\right) = \frac{3a^2b + 3ab^2 + 2\sqrt{ab}\,(a^2+b^2) - a^3 - b^3}{(a+b)(\sqrt{a}\,+\sqrt{b}\,)^2} < 0$$

$${}^{i}SMR(a\,,b) - H_e(a\,,b) = \frac{4ab}{(\sqrt{a}\,+\sqrt{b}\,)^2} - \left(\frac{a+\sqrt{ab}+b}{3}\right) = \frac{-a^2 - b^2 + 8ab - 3\sqrt{ab}\,(a+b)}{3(\sqrt{a}\,+\sqrt{b}\,)^2} < 0$$

$${}^{i}SMR(a\,,b) - RSM(a\,,b) = \frac{4ab}{(\sqrt{a}\,+\sqrt{b}\,)^2} - \sqrt{\frac{a^2+b^2}{2}} = \frac{4\sqrt{2}ab - a\sqrt{a^2+b^2} - b\sqrt{a^2+b^2} - 2\sqrt{ab(a^2+b^2)}}{\sqrt{2}(\sqrt{a}\,+\sqrt{b}\,)^2} < 0$$

Combining the above, the following inequality chain is obtained

$H < {}^{i}SMR(a\,,b) < G < A < C < H_e < RSM.$

Theorem 5.3: The square mean root and G – Complementary to square mean root are related by the inequality $a < {}^{i}SMR(a\,,b) < SMR(a\,,b) < b$.

Proof: For $a < b$, consider $SMR(a\,,b) - {}^{i}SMR(a\,,b) = \left(\frac{\sqrt{a}+\sqrt{b}}{2}\right)^2 - \frac{4ab}{(\sqrt{a}+\sqrt{b})^2}$

$$= \frac{(\sqrt{a}+\sqrt{b})^4 - 16ab}{2(\sqrt{a}+\sqrt{b})^2} > 0$$

Combining this inequality with lemma (4.1), gives $a < {}^{i}SMR(a\,,b) < SMR(a\,,b) < b$.

Hence the proof of the theorem.

Theorem 5.4: The Inverse Square Mean Root is Convex with respect to Arithmetic Mean, $\forall\, a \le b \le c$.

Proof:

Consider, ${}^{i}SMR(a\,,b) = \frac{4ab}{(\sqrt{a}+\sqrt{b})^2}$ and $A(a,b) = \frac{a+b}{2}$

If $a = a\,,\ b = 1$ then $f(a) = \frac{4a}{(\sqrt{a}+1)^2}$, $g(a) = \frac{a+1}{2}$ then by lemma 2.5,

$$\Delta = \begin{vmatrix} 1 & f(a) & g(a) \\ 1 & f(b) & g(b) \\ 1 & f(c) & g(c) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & f(a) & g(a) \\ 0 & f(b)-f(a) & g(b)-g(a) \\ 0 & f(c)-f(a) & g(c)-g(a) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \frac{4a}{(\sqrt{a}+1)^2} & \frac{a+1}{2} \\ 0 & \frac{4b}{(\sqrt{b}+1)^2} - \frac{4a}{(\sqrt{a}+1)^2} & \left(\frac{b+1}{2}\right) - \left(\frac{a+1}{2}\right) \\ 0 & \frac{4c}{(\sqrt{c}+1)^2} - \frac{4a}{(\sqrt{a}+1)^2} & \left(\frac{c+1}{2}\right) - \left(\frac{a+1}{2}\right) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \frac{4a}{(\sqrt{a}+1)^2} & \frac{a+1}{2} \\ 0 & \frac{4b(\sqrt{a}+1)^2 - 4a(\sqrt{b}+1)^2}{(\sqrt{b}+1)^2(\sqrt{a}+1)^2} & \left(\frac{b-a}{2}\right) \\ 0 & \frac{4c(\sqrt{a}+1)^2 - 4a(\sqrt{c}+1)^2}{(\sqrt{c}+1)^2(\sqrt{a}+1)^2} & \left(\frac{c-a}{2}\right) \end{vmatrix}$$

$$= 1\left[\frac{4b(\sqrt{a}+1)^2 - 4a(\sqrt{b}+1)^2}{(\sqrt{b}+1)^2(\sqrt{a}+1)^2}\right]\left(\frac{c-a}{2}\right) - \left[\frac{4c(\sqrt{a}+1)^2 - 4a(\sqrt{c}+1)^2}{(\sqrt{c}+1)^2(\sqrt{a}+1)^2}\right]\left(\frac{b-a}{2}\right)$$

$$= \frac{1}{8}\left[\frac{(4b-4)(c-1)}{(\sqrt{b}+1)^2} - \frac{(4c-4)(b-1)}{(\sqrt{c}+1)^2}\right]$$

$$= \frac{(b-1)(c-1)}{2}\left[\frac{1}{(\sqrt{b}+1)^2} - \frac{1}{(\sqrt{c}+1)^2}\right] > 0.$$

Hence the theorem.

The graph below represents the convexity of ${}^{i}SMR$ with respect to AM

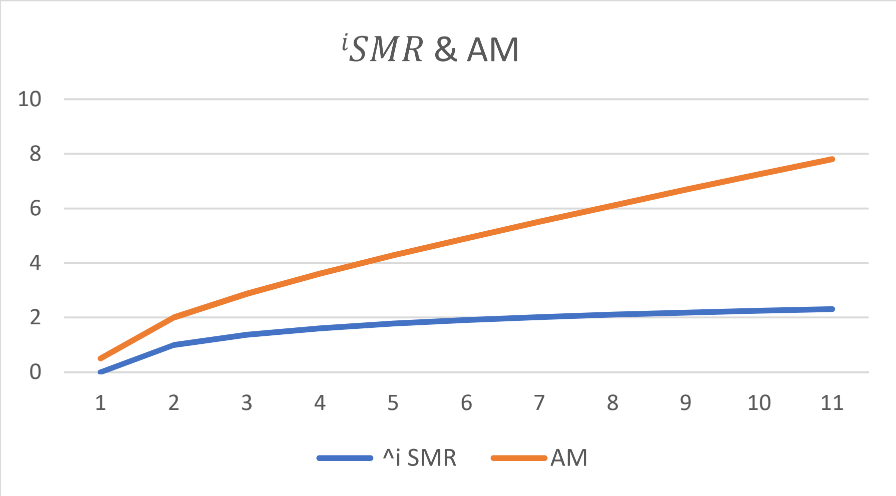

Figure 5.1: Convexity of ${}^{i}SMR$ with respect to AM

From the figure 5.1, the graph of AM lies above every chord (line segment) connecting any two points on the curve when those points are compared based on the corresponding values of ${}^{i}SMR$. Also, ${}^{i}SMR$ shows the convex curvature with respect to AM and hence it is clear that ${}^{i}SMR$ is convex with respect to AM.

Theorem 5.5: The Inverse Square Mean Root is Concave with respect to Geometric Mean, $\forall\, a \le b \le c$.

Proof: Consider, ${}^{i}SMR(a\,, b) = \frac{4ab}{(\sqrt{a}\,+\sqrt{b}\,)^2}$ and $G(a,b) = \sqrt{ab}$

If $a = a\,,\ b = 1$ then $f(a) = \frac{4a}{(\sqrt{a}\,+1)^2}$, $g(a) = \sqrt{a}$ then by lemma 2.5,

$$\Delta = \begin{vmatrix} 1 & f(a) & g(a) \\ 1 & f(b) & g(b) \\ 1 & f(c) & g(c) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & f(a) & g(a) \\ 0 & f(b)-f(a) & g(b)-g(a) \\ 0 & f(c)-f(a) & g(c)-g(a) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \frac{4a}{(\sqrt{a}\,+1)^2} & \sqrt{a} \\ 0 & \frac{4b}{(\sqrt{b}\,+1)^2} - \frac{4a}{(\sqrt{a}\,+1)^2} & (\sqrt{b}) - (\sqrt{a}) \\ 0 & \frac{4c}{(\sqrt{c}\,+1)^2} - \frac{4a}{(\sqrt{a}\,+1)^2} & (\sqrt{c}) - (\sqrt{a}) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \frac{4a}{(\sqrt{a}\,+1)^2} & \sqrt{a} \\ 0 & \frac{4b(\sqrt{a}\,+1)^2 - 4a(\sqrt{b}\,+1)^2}{(\sqrt{b}\,+1)^2(\sqrt{a}\,+1)^2} & (\sqrt{b}) - (\sqrt{a}) \\ 0 & \frac{4c(\sqrt{a}\,+1)^2 - 4a(\sqrt{c}\,+1)^2}{(\sqrt{c}\,+1)^2(\sqrt{a}\,+1)^2} & (\sqrt{c}) - (\sqrt{a}) \end{vmatrix}$$

$$= 1\left[\frac{4b(\sqrt{a}\ +1)^2 - 4a(\sqrt{b}\ +1)^2}{(\sqrt{b}\ +1)^2(\sqrt{a}\ +1)^2}\right]\left[\sqrt{c} - \sqrt{a}\right] - \left[\frac{4c(\sqrt{a}\ +1)^2 - 4a(\sqrt{c}\ +1)^2}{(\sqrt{c}\ +1)^2(\sqrt{a}\ +1)^2}\right]\left[\sqrt{b} - \sqrt{a}\right]$$

$$= \frac{1}{4}\left[\frac{(4b-4)(\sqrt{c}-1)}{(\sqrt{b}\ +1)^2} - \frac{(4c-4)(\sqrt{b}-1)}{(\sqrt{c}\ +1)^2}\right]$$

$$= -\frac{(\sqrt{b}-1)(\sqrt{c}-1)(\sqrt{b}-\sqrt{c})}{(\sqrt{b}+1)(\sqrt{c}+1)} < 0.$$

Hence the theorem.

The graph below represents the convexity of ${}^{i}SMR$ with respect to GM

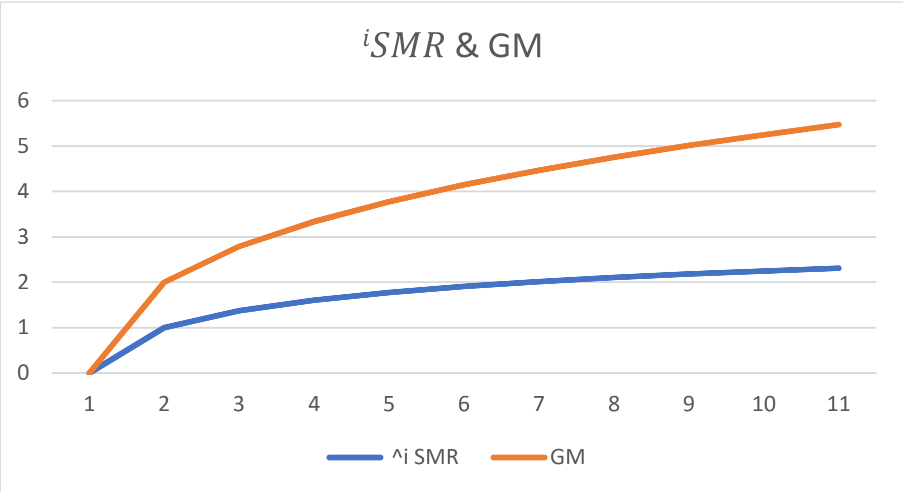

Figure 5.2: Convexity of ${}^{i}SMR$ with respect to GM

From the figure 5.2, the graph of GM lies above every chord (line segment) connecting any two points on the curve when those points are compared based on the corresponding values of ${}^{i}SMR$. ${}^{i}SMR$ shows concave curvature with respective to GM and hence it is clear that ${}^{i}SMR$ is concave with respect to GM.

Theorem 5.6: The Inverse Square Mean Root is Convex with respect to Harmonic Mean, $\forall\, a \le b \le c$.

Proof: Consider, ${}^{i}SMR(a\,,b)=\frac{4ab}{(\sqrt{a}\ +\sqrt{b}\,)^2}$ and $H(a,b) = \frac{2ab}{a+b}$

If $a = a\,,\ b = 1$ then $f(a) = \frac{4a}{(\sqrt{a}\ +1)^2}$, $g(a) = \frac{2a}{a+1}$ then by lemma 2.5,

$$\Delta = \begin{vmatrix} 1 & f(a) & g(a) \\ 1 & f(b) & g(b) \\ 1 & f(c) & g(c) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & f(a) & g(a) \\ 0 & f(b) - f(a) & g(b) - g(a) \\ 0 & f(c) - f(a) & g(c) - g(a) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \frac{4a}{(\sqrt{a}+1)^2} & \frac{2a}{a+1} \\ 0 & \frac{4b}{(\sqrt{b}+1)^2} - \frac{4a}{(\sqrt{a}+1)^2} & \frac{2b}{b+1} - \frac{2a}{a+1} \\ 0 & \frac{4c}{(\sqrt{c}+1)^2} - \frac{4a}{(\sqrt{a}+1)^2} & \frac{2c}{c+1} - \frac{2a}{a+1} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \frac{4a}{(\sqrt{a}+1)^2} & \frac{2a}{a+1} \\ 0 & \frac{4b(\sqrt{a}+1)^2 - 4a(\sqrt{b}+1)^2}{(\sqrt{b}+1)^2(\sqrt{a}+1)^2} & 2(b-a) \\ 0 & \frac{4c(\sqrt{a}+1)^2 - 4a(\sqrt{c}+1)^2}{(\sqrt{c}+1)^2(\sqrt{a}+1)^2} & 2(c-a) \end{vmatrix}$$

$$= 2\left[\frac{4b(\sqrt{a}+1)^2 - 4a(\sqrt{b}+1)^2}{(\sqrt{b}+1)^2(\sqrt{a}+1)^2}\right](c-a) - \left[\frac{4c(\sqrt{a}+1)^2 - 4a(\sqrt{c}+1)^2}{(\sqrt{c}+1)^2(\sqrt{a}+1)^2}\right](b-a)$$

$$= \frac{1}{2}\left[\frac{(4b-4)(c-1)}{(\sqrt{b}+1)^2} - \frac{(4c-4)(b-1)}{(\sqrt{c}+1)^2}\right]$$

$$= 2(b-1)(c-1)\left[\frac{1}{(\sqrt{b}+1)^2} - \frac{1}{(\sqrt{c}+1)^2}\right] > 0.$$

Hence the theorem.

The graph below represents the convexity of ${}^{i}SMR$ with respect to HM.

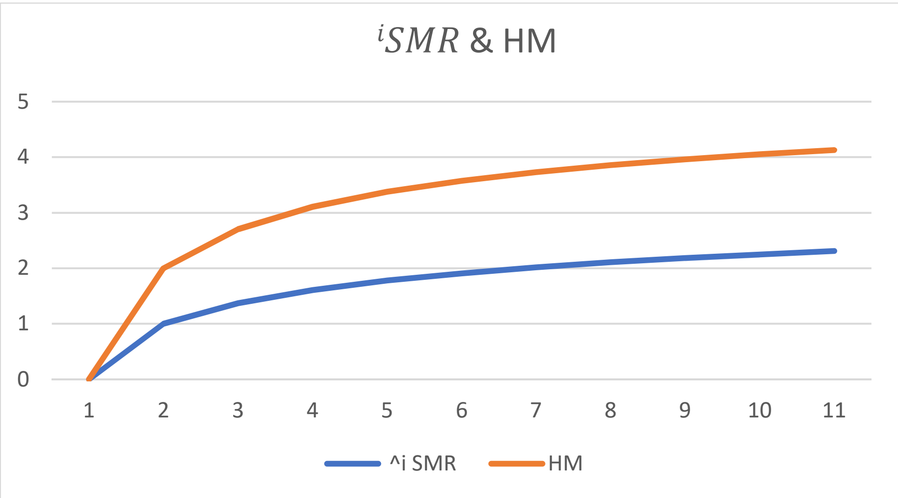

Figure 5.3: The convexity of ${}^{i}SMR$ with respect to HM

From the figure 5.3, the graph of HM lies above every chord (line segment) connecting any two points on the curve when those points are compared based on the corresponding values of ${}^{i}SMR$. ${}^{i}SMR$ shows convex curvature with respective to HM and hence it is clear that ${}^{i}SMR$ is convex with respect to HM.

Theorem 5.7: The Inverse Square Mean Root is Concave with respect to Contra−Harmonic Mean, $\forall\, a \le b \le c.$

Proof: Consider, ${}^{i}SMR(a\,,b) = \frac{4ab}{(\sqrt{a}+\sqrt{b})^2}$ and $C(a,b) = \frac{a^2+b^2}{a+b}$

If $a = a$, $b = 1$ then $f(a) = \frac{4a}{(\sqrt{a}+1)^2}$, $g(a) = \frac{a^2+1}{a+1}$ then by lemma 2.5,

$$\Delta = \begin{vmatrix} 1 & f(a) & g(a) \\ 1 & f(b) & g(b) \\ 1 & f(c) & g(c) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & f(a) & g(a) \\ 0 & f(b)-f(a) & g(b)-g(a) \\ 0 & f(c)-f(a) & g(c)-g(a) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \frac{4a}{(\sqrt{a}+1)^2} & \frac{a^2+1}{a+1} \\ 0 & \frac{4b}{(\sqrt{b}+1)^2} - \frac{4a}{(\sqrt{a}+1)^2} & \left(\frac{b^2+1}{b+1}\right) - \left(\frac{a^2+1}{a+1}\right) \\ 0 & \frac{4c}{(\sqrt{c}+1)^2} - \frac{4a}{(\sqrt{a}+1)^2} & \left(\frac{c^2+1}{c+1}\right) - \left(\frac{a^2+1}{a+1}\right) \end{vmatrix}$$

$$= \begin{vmatrix} 1 & \frac{4a}{(\sqrt{a}+1)^2} & \frac{a^2+1}{a+1} \\ 0 & \frac{4b(\sqrt{a}+1)^2 - 4a(\sqrt{b}+1)^2}{(\sqrt{b}+1)^2(\sqrt{a}+1)^2} & \left(\frac{b^2+1}{b+1}\right) - \left(\frac{a^2+1}{a+1}\right) \\ 0 & \frac{4c(\sqrt{a}+1)^2 - 4a(\sqrt{c}+1)^2}{(\sqrt{c}+1)^2(\sqrt{a}+1)^2} & \left(\frac{c^2+1}{c+1}\right) - \left(\frac{a^2+1}{a+1}\right) \end{vmatrix}$$

$$= \left[\frac{4b(\sqrt{a}+1)^2 - 4a(\sqrt{b}+1)^2}{(\sqrt{b}+1)^2(\sqrt{a}+1)^2}\right]\left[\left(\frac{c^2+1}{c+1}\right) - \left(\frac{a^2+1}{a+1}\right)\right]$$

$$- \left[\frac{4c(\sqrt{a}+1)^2 - 4a(\sqrt{c}+1)^2}{(\sqrt{c}+1)^2(\sqrt{a}+1)^2}\right]\left[\left(\frac{b^2+1}{b+1}\right) - \left(\frac{a^2+1}{a+1}\right)\right] < 0.$$

Hence the theorem.

The graph below represents the Convexity of ${}^{i}SMR$ with respect to CHM

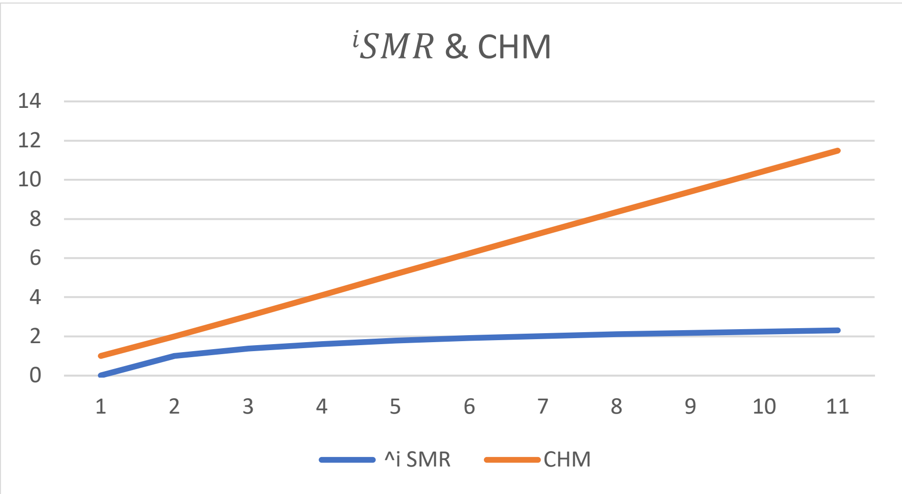

Figure 5.4: Convexity of ${}^{i}SMR$ with respect to CHM

From the figure 5.4, the graph of CHM lies above every chord (line segment) connecting any two points on the curve when those points are compared based on the corresponding values of ${}^{i}SMR$ and shows concave curvature with respective to CHM and hence it is clear that ${}^{i}SMR$ is concave with respect to CHM.

Theorem 5.8: For $a\,,b>0$, the G – Complementary to Square mean root is Schur Convex.

Proof: The G – Complementary to Square mean root is given by ${}^{i}SMR(a\,,b)=\frac{4ab}{(\sqrt{a}\;+\sqrt{b})^{2}}$

On differentiating partially with respect to a and b,

$$\frac{\partial({}^{i}SMR)}{\partial a}=\frac{4b^2\;+4b\sqrt{ab}}{(\sqrt{a}\;+\sqrt{b})^4} \qquad \text{and} \qquad \frac{\partial({}^{i}SMR)}{\partial b}=\frac{4a^2\;+4a\sqrt{ab}}{(\sqrt{a}\;+\sqrt{b})^4}$$

$$\frac{\partial({}^{i}SMR)}{\partial a}-\frac{\partial({}^{i}SMR)}{\partial b}=\left(\frac{4b^2\;+4b\sqrt{ab}}{(\sqrt{a}\;+\sqrt{b})^4}\right)-\left(\frac{4a^2\;+4a\sqrt{ab}}{(\sqrt{a}\;+\sqrt{b})^4}\right)$$

$$(a-b)\left(\frac{\partial({}^{i}SMR)}{\partial a}-\frac{\partial({}^{i}SMR)}{\partial b}\right)=(a-b)\left[\left(\frac{4b^2\;+4b\sqrt{ab}}{(\sqrt{a}\;+\sqrt{b})^4}\right)-\left(\frac{4a^2\;+4a\sqrt{ab}}{(\sqrt{a}\;+\sqrt{b})^4}\right)\right]>0$$

Hence the G – Complementary Square mean root is Schur Convex.

Theorem 5.9: For $a\,,b>0$, the G – Complementary to Square mean root is Schur Geometric Convex.

Proof: The G – Complementary to Square mean root is given by ${}^{i}SMR(a\,,b)=\frac{4ab}{(\sqrt{a}\;+\sqrt{b})^{2}}$

On differentiating partially with respect to a and b,

$$\frac{\partial({}^{i}SMR)}{\partial a}=\frac{4b^2\;+4b\sqrt{ab}}{(\sqrt{a}\;+\sqrt{b})^4} \qquad \text{and} \qquad \frac{\partial({}^{i}SMR)}{\partial b}=\frac{4a^2\;+4a\sqrt{ab}}{(\sqrt{a}\;+\sqrt{b})^4}$$

$$\left(a\frac{\partial({}^{i}SMR)}{\partial a}-b\frac{\partial({}^{i}SMR)}{\partial b}\right)=a\left(\frac{4b^2\;+4b\sqrt{ab}}{(\sqrt{a}\;+\sqrt{b})^4}\right)-b\left(\frac{4a^2\;+4a\sqrt{ab}}{(\sqrt{a}\;+\sqrt{b})^4}\right)=\left(\frac{4ab\,(b-a)}{(\sqrt{a}\;+\sqrt{b})^4}\right)$$

$$(lna-lnb)\left(a\frac{\partial({}^{i}SMR)}{\partial a}-b\frac{\partial({}^{i}SMR)}{\partial b}\right)=(lna-lnb)\left[\left(\frac{4ab\,(b-a)}{(\sqrt{a}\;+\sqrt{b})^4}\right)\right]\geq 0,\text{ for all } a\,,b>0$$

Hence the G – Complementary Square mean root is Schur Geometric Convex.

Theorem 5.10: For $a\,,b>0$, the G – Complementary to Square mean root is Schur Harmonic Convex.

Proof: The G – Complementary to Square mean root is given by ${}^{i}SMR(a\,,b)=\frac{4ab}{(\sqrt{a}\;+\sqrt{b})^{2}}$

On differentiating partially with respect to a and b,

$$\frac{\partial({}^{i}SMR)}{\partial a}=\frac{4b^2\;+4b\sqrt{ab}}{(\sqrt{a}\;+\sqrt{b})^4} \qquad \text{and} \qquad \frac{\partial({}^{i}SMR)}{\partial b}=\frac{4a^2\;+4a\sqrt{ab}}{(\sqrt{a}\;+\sqrt{b})^4}$$

$$\left(a^2\frac{\partial({}^{i}SMR)}{\partial a}-b^2\frac{\partial({}^{i}SMR)}{\partial b}\right)=a^2\left(\frac{4b^2\;+4b\sqrt{ab}}{(\sqrt{a}\;+\sqrt{b})^4}\right)-b^2\left(\frac{4a^2\;+4a\sqrt{ab}}{(\sqrt{a}\;+\sqrt{b})^4}\right)=\left[\left(\frac{4(ab)^{\frac{1}{3}}\,(a-b)^2}{(\sqrt{a}\;+\sqrt{b})^4}\right)\right]>0$$

Hence the G – Complementary Square mean root is Schur Harmonic Convex.

6. Conclusion

In this article, it is justified that the invariant square mean root denoted by ${}^{i}SMR$ is a mean. It is also established an inequality chain involving classical means. Also, verified standard convexity properties involving ${}^{i}SMR$.

REFERENCES:

[1] P. S. Bullen "Handbook of means and their inequalities", Kluwer Acad. Publ., Dordrecht, 2003.
[2] V. Lokesha and K. M. Nagaraja, Relation between series and important means, Advances in Theoretical and Applied Mathematics, 2(1), (2007), pp. 31 – 36.
[3] Silvia. Toader and Gheorghe. Toader, Complementary of a Greek mean with respect to another, Automat. Computer Appl. Math., 13 (1), (2004), pp. 203 – 208.
[4] V. Lokesha, Zhi-Hua Zhang and K. M. Nagaraja, Gnan means for two variables, Far East Journal of Applied Mathematics, 31(2), (2008), pp. 263 – 272.
[5] V. Lokesha, K. M. Nagaraja and Y. Simsek, New Inequalities on the homogeneous functions, J. Indone. Math. Soc., 15(1), (2009), pp. 49 – 59.
[6] R Sampath Kumar, K. M. Nagaraja and G Narasimhan, New Family of Heinz Type Means, International Journal of Pure and Applied Mathematics, 118 (18), (2018), pp. 3427 – 3433.
[7] Sandeep Kumar, V. Lokesha, U.K. Misra and K.M. Nagaraja, Power Type-Centroidal Mean and Its Dual, Gen. Math. Notes, 25(1), (2015), pp. 33 – 42.
[8] Silvia. Toader and Gheorghe. Toader, Greek means and Arithmetic and Geometric means, RGMIA monograph, Australia, 2005. http://rgmia.vu.edu.au/html
[9] V. Lokesha, K. M. Nagaraja, S. Padmanabhan and Naveen Kumar. B, Oscillatory type means in Greek means, Int. e-Journal of Engg. Maths. Theory and Applications, 9, (2010), 18 – 26.
[10] V. Lokesha, Zhi-Hua Zhang and K. M. Nagaraja, r^{th} Oscillatory mean for several positive arguments, Ultra Scientist, 18(3) M, (2006), pp. 519 – 522.
[11] V. Lokesha, Padmanabhan. S, K. M. Nagaraja and Y. Simsek, Relation between Greek Means and Various means, General Mathematics, 17(3), (2009), pp. 3 – 13.
[12] Vimala T, K M Nagaraja, and Sampath Kumar R, A Harmonic and Heron Mean Inequalities for arguments in different intervals, International Journal of Mathematical Archive – 10(7), (2019), pp. 47 – 52.
[13] K. M. Nagaraja and P. S. K. Reddy, A Note on power mean and generalized contra harmonic mean, Scientia Magna, 8(3), (2012), pp. 60 – 62.
[14] K. Murali, and K. M. Nagaraja, Schur Convexity of Stolarsky's Extended Mean Values, Journal of Mathematical Inequalities, JMI-10-59,10(3), (2016), pp.725 – 735.
[15] Lakshmi Janardhana. R. C, K. M. Nagaraja and Narasimhan. G. A note on G – Complementary to Heron Mean, International Journal of Research and Analytical Reviews, 5(3), (2018), pp. 369 – 373.
[16] V. Lokesha, Naveen Kumar B, K. M. Nagaraja, Abdel Mejid Bayad and M. Saraj, New Means and its Properties, Proceedings of Jang Jeon Mathematical Society, 14(3), (2010), 243 – 254.
[17] E. Dhanalakshmi, G. Narasimhan, K. M. Nagaraja, R. Sampath Kumar (2023) Schur convexity of complementary geometric mean, Journal of Propulsion Technology, 44(2), pp.1327 – 1338.
[18] Nagaraja, K.M, Sampathkumar, R., Chethan Kumar, G.D., Dhanya, P. (2023). Schur convexities of ratio of one parameter power exponential mean and its invariant, Proceedings of the Jang Jeon Mathematical Society, 26(2), pp. 161–168.

[19] Nagaraja, K. M., Hejib, A. M., &Venkataramana, B. S. (2021). Complementary of Classical Means with Respect to Heron Mean and Their Schur Convexities. *Proceedings on Engineering Sciences*, *3*(2), 231 – 238.
[20] K. M. Nagaraja, Murali K, and Lakshmi Janardhana R C, Improvement of Harmonic and Contra Harmonic Mean Inequality Chain, International Journal of Pure and Applied Mathematics, 114(4), (2017), pp.771 – 776.
[21] K. M. Nagaraja, P. S. K. Reddy and Naveen Kumar. B, Refinement of Inequality Involving Ratio of Means for Four Positive Arguments, Bulletin of International Mathematical Virtual Institute, 3, (2013), pp. 135 – 138.
[22] Chinni Krishna. R, K.M. Nagaraja, Sampathkumar. R. and P. Siva Kota Reddy, (2023). The Ky Fan type inequalities for the means whose arguments lying on linear and curved path, Eur. Chem. Bull. 12(10), pp. 2338 – 2347.