ON PROXIMITY NUMBER OF A GRAPH

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ABSTRACT

A Proximity set S of a graph G is a Split Proximity set if the induced subgraph $< V - S >$ is disconnected. The split-proximity number $n_s(G)$ is the minimum cardinality of a split-proximity set. In this paper, we have obtained bounds for $n_s(G)$ in terms of order, size and other parameters of graphs.

Keywords: Domination number, Split Domination Number, Split-Proximity.

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1. INTRODUCTION

The graphs considered here are finite, undirected, without loops or multiple edges and connected. Unless otherwise stated, all graphs are assumed to have 'p' vertices and 'q' edges.

A set S of vertices in graph G is a Proximity set $(n - set)$ of G if $G = \underset{u \in S}{\cup} < N(u) >$, where $< N(u) >$ is the subgraph induced by u and all vertices adjacent to $u \in S$, $S\{u\}$ is not Proximity set of G. The Proximity number $n_\circ(G)$ of G is a minimum cardinality of a $n - set$ of G. This parameter is introduced by E. Sampathkumar and P. S. Neeralagi [6].

There are many types of domination numbers in literature [2]. Similarly we can define different types of Proximity numbers by imposing certain conditions on Proximity sets and derive some of the properties.

A Proximity set S is said to be a maximal Proximity set of G if the induced subgraph $< V - S >$ is not a Proximity set of G. The maximal Proximity number $n_m(G)$ of G is the minimum cardinality of a maximal Proximity set of G. This parameter is introduced by N.D. Soner et al [6].

In this chapter, we introduce the concept of SplitProximity as follows :

A Proximity set S of a graph G is a Split Proximity set if the induced subgraph $< V - S >$ is disconnected. The SplitProximity number $n_s(G)$ is the minimum cardinality of a SplitProximity set.

Thus, we observe that for any graph G,

$\gamma(G) \leq n_\circ(G) \leq n_s(G) \leq \alpha_\circ(G)$..........................(I)

$\gamma(G) \leq \gamma_S(G) \leq n_s(G) \leq \alpha_\circ(G)$...........................(II)

Now we will prove the following results.

2. RESULTS

Theorem A [4] A dominating set D of G is a Split dominating set if and only if there exists two vertices $w_1, w_2 \in V - D$ such that $w_1 - w_2$ path contains a vertex of D.

Theorem 2.1 For any graph G, $n_\circ(G) \leq n_s(G)$..........................(1)

Further the bound is attained if and only if there exists two vertices $w_1, w_2 \in V - S$ such that every $w_1 - w_2$ path contains a vertex of S where S is a $n_\circ - set$ of G.

Proof: Equation (1) follows from the definition of SplitProximity set.

Further let S be a Proximity set such that there exists two vertices $w_1, w_2 \in V - S$ such that every $w_1 - w_2$ path contains a vertex of S. Then $< V - S >$ is disconnected. Hence S is a SplitProximity set. This implies $n_s(G) \leq n_\circ(G)$. Then from (1) we have $n_\circ(G) = n_s(G)$.

Conversely suppose the bound is attained. Then if S is a Proximity set, it is also a SplitProximity set. This implies $< V - S >$ is disconnected. Hence there exist two vertices $w_1, w_2 \in V - S$ such that every $w_1 - w_2$ path contains a vertex S.

Theorem B [6] : For a graph G, $n_\circ(G) = \gamma(G)$ if and only if there exists a minimum dominating set S. Such that every line in $< V - S >$ belongs to $< N(u) >$ for some $u \in D$.

Theorem 2.2 For any graph G,

$$\gamma_S(G) \leq n_s(G) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

Further the bound is attained if and only if there exists a minimum Split dominating set S such that every line in $< V - S >$ belongs to $< N(u) >$ for some $u \in S$.

Proof : Since every SplitProximity set is a Split dominating set, hence Split dominating number is less than SplitProximity number. Suppose the bound is attained. This implies the condition is satisfied from Theorem 5.A [4].

Conversely, suppose that given condition is satisfied for some Split dominating set S. Then again by Theorem 5.B [6], S is a Proximity set. Since $< V - S >$ is disconnected. S is a Split Proximity set and hence from (2) the bound is attained.

Theorem C [6] For any graph G without isolated points,

$$\gamma(G) \leq n_\circ(G) \leq \alpha_\circ(G)$$

Theorem 2.3 For any graph G without isolated points,

$$n_s(G) \leq \alpha_\circ(G) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

Further the bound is attained if and only if there exist a Split Proximity set S of G for which $V - S$ is independent with at least two vertices.

Proof : Let S be vertex cover of G. Then, $V - S$ is independent with at least two vertices. This implies, $< V - S >$ is disconnected. Also S is a Proximity set from Theorem 5.C [6]. Hence S is a Split Proximity set of G. This proves that the Split Proximity number is less than or equal to vertex covering number.

Now to prove the second part, suppose there exist a Split Proximity set S of G for which $V - S$ is independent with at least two vertices. This implies S is a vertex cover of G. Thus vertex covering number of G is less than or equal to the cardinality of S. Hence from (3), the bound is attained.

Conversely, suppose equality holds. Then there exists a Split Proximity set S which is a vertex cover with $|S| = \alpha_\circ(G)$. Then obviously $V - S$ is independent with at least two vertices.

Theorem D [4] For any graph G, $\gamma \leq \gamma_s$

Hence from Theorem 5.1, 5.2, 5.3, 5.C [6] and 5.D [4]

we have,

$$\gamma(G) \leq n_\circ(G) \leq n_s(G) \leq \alpha_\circ(G) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(I)}$$

$$\gamma(G) \leq \gamma_s(G) \leq n_s(G) \leq \alpha_\circ(G) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(II)}$$

Theorem 2.4 For any graph G,

$$k(G) \leq n_s(G) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(4)}$$

Where $k(G)$ is the connectivity of graph G.

Proof : Let S be a Split Proximity set of G. Then $< V - S >$ is disconnected.

Hence $k(G) \leq n_s(G)$

Next, we list the exact value of $n_s(G)$ for some standard graphs

Theorem 2.5 (i) For a path P_n with n vertices,

$$n_s(P_n) - \left\lceil \frac{n}{3} \right\rceil \qquad n \geq 3 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(5)}$$

(ii) For a circle C_n with n vertices,

$$n_s(C_n) - \left\lceil \frac{n}{2} \right\rceil \qquad n \geq 4 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(6)}$$

(iii) For a wheel W_n with n vertices,

$n_s(W_n) = 3$ $n \geq 5$…….…………..……..(7)

(iv) For a bipartite graph, without isolates, with bipartition $\{v_1, v_2\}$

of V(G),

$n_s(G) \leq \min\{|v_1|, |v_2|\}$………………………………….(8)

Moreover the bound is attained by the graphs $K_{m,n}$

Proof :

(i)For a path P_n with n vertices where $n \geq 3$, every Proximity set is a Split Proximity set. Hence (5) follows.

(ii) For a cycle C_n with n vertices where $n \geq 4$, every Proximity set is a Split Proximity set. Hence (6) follows.

(iii) For a wheel W_n with n vertices where $n \geq 5$, the vertex with degree $p - 1$ together with two non adjacent vertices on the cycle form a Split Proximity set. Hence (7) follows.

(iv) For a bipartite graph with bipartition $\{V_1, V_2\}$ of $V(G)$, both the sets with cardinality V_1 and V_2 are Split Proximity sets. Hence (8) follows. Further if it is a complete bipartite graph then equality holds since for any V_i, i=1,2,3,…….

$V_i - \{u\}$ is not a Split Proximity set.

Theorem E [6] For any bipartite graph G without isolated points,

$$n_\circ(G) = \alpha_\circ(G) = \beta_1(G)$$

Theorem 2.6 For any bipartite graph G without isolated points,

$n_\circ(G) = n_s(G) = \alpha_\circ(G) = \beta_1(G)$………………………(9)

Proof : This follows from Theorem 5.E [6] and Result (I)

Theorem 2.7 A Split Proximity set S is minimal if and only if for each vertex $v \in S$, one of the following conditions is satisfied

(i) v is an isolate in $< S >$

(ii) There exist a vertex $u \in V - S$ adjacent to v but not adjacent to any vertex $w \in S$ adjacent to v.

(iii) $< (V - S) \cup \{v\} >$ is connected.

Proof: Suppose S is minimal, on the contrary, if there exists $v \in S$ such that v does not satisfy any of the given conditions. Then $S' = S - \{v\}$ is a Proximity set of G from (i) and (ii) and $< V - S' >$ is disconnected from (iii) This implies S' is Split Proximity set of G. This is a contradiction. This proves that necessity.

Sufficiency is straight forward.

Theorem F [1] : For any non trivial connected graph G,

$$\alpha_\circ(G)+\beta_\circ(G) = p$$

Theorem 2.8:

i) For any graph G,

$$\gamma(G) \leq n_\circ(G) \leq n_s(G) \leq (\chi(G) - 1)\beta_\circ(G)\ldots\ldots\ldots\ldots\ldots\ldots\ldots (10)$$

Provided $\chi(G) \geq 2$, where $\chi(G)$ is the chromatic number of graph G.

ii) If G is bipartite graph which is not totally disconnected, Then,

$$\gamma(G) \leq n_\circ(G) \leq n_s(G) \leq \beta_\circ(G) \leq \chi(\bar{G})\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (11)$$

Where $\bar{G}$ is complement of G.

Proof : Here we need to establish only the upper bound since lower bounds from I.

From Theorem 5.F [1] and the fact that $p \leq \chi(G)(\beta_\circ(G))$

(See [1]) we have,

$$p - \beta_\circ(G) \leq \beta_\circ(G)(\chi(G) - 1)$$

i.e. $\alpha_\circ(G) \leq \beta_\circ(G)(\chi(G) - 1)$

Hence (10) follows from (1) and the fact that $\alpha_\circ(G) \leq \beta_\circ(G)(\chi(G) - 1)$

If G is bipartite, $\chi(G) = 2$. Also (10) implies $n_s(G) \leq \beta_\circ(G)$

Hence (11) follows from the facts that $n_s(G) \leq \beta_\circ(G)$ and $\beta_\circ(G) \leq \chi(\bar{G})$ (See [1]).

Theorem 2.9 For any graph G,

$$n_s(G) = 1 \text{..................................(12)}$$

If and only if there exits a cut vertex with degree $p - 1$

Proof : Suppose v is cutvertex of G of degree $p - 1$, then $\{v\}$ is a Proximity set. Further since $< V - \{v\} >$ is disconnected. This implies $\{v\}$ is a Split Proximity set. Hence $n_s(G) = 1$

Conversely, suppose $n_s(G) = 1$. Then, obviously there exists a cutvertex which is adjacent to all vertices. Hence there exists a cutvertex with degree $p - 1$.

Theorem G [6] For any (p, q) graph G,

$$p - q + q_\circ \leq n_\circ(G) \leq p - \Delta(G)$$

$$\left\lceil \frac{p}{\Delta(G) + 1} \right\rceil \leq n_\circ(G) \leq p - \beta_\circ(G) + p_\circ$$

Where $q_\circ$ =minimum {q(<D>;D is a minimal dominating set of G}

$p_\circ$ =the number of isolated vertices in G,

$\beta_\circ$ =set of independent vertices in G.

Theorem 2.10 For any connected (p, q) graph G,

$$p - q + q_\circ \leq n_s(G) \text{............................(13)}$$

$$\left\lceil \frac{p}{\Delta(G)+1} \right\rceil \leq n_s(G) \leq p - \beta_\circ(G) \ldots\ldots\ldots\ldots\ldots (14)$$

Proof : The lower bounds in (13) and (14) follow from (1) and Theorem 5.G [6]. To prove upper bound in (14), we observe that $(V - M)$ is a Split Proximity set where M is the set of $\beta_\circ$ independent points of G.

The lower bound in (13) and (14) is attained for the following graph in Figure 5

The upper bound in (14) is attained for any tree

The lower bound in (14) is attained by the following graph in figure 6.

Theorem 2.11

(i) $n_s(G) > p - \Delta(G)$ if there exist a non-cutvertex of degree $p - 1$

(ii) $n_s(G) \leq p - \Delta(G)$ if G has no triangle.

Proof :

(i) Let G has a non-cutvertex v of degree $p - 1$. Then $\Delta(G) = p - 1$. Since v is the non-cutvertex, $n_s(G) \geq 2$. Hence $n_s(G) > p - \Delta(G)$.

(ii) If G has no triangle then $n_s(G) \leq p - \Delta(G)$ from (9) and Theorem 5.G [6].

Now we obtain a Nordhaus-Gaddum type result.

Theorem 2.12 Let G be a graph such that both G and $\bar{G}$ are connected, then

$$n_s(G) + n_s(\bar{G}) \leq p(p - 3) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15)$$

Further the bound is attained if and only if $G = P_4$

Proof : We have $n_s(G) \leq \alpha_\circ(G)$ from (3).

Since both G and $\bar{G}$ are connected, $\Delta(G), \Delta(\bar{G}) < p - 1$

This implies $\beta_\circ(G), \beta_\circ(\bar{G}) \geq 2$.

Hence $n_s(G) \leq p - 2$

$= 2(p-1) - p$

$\leq (2q - p)$

Similarly $n_s(\bar{G}) \leq 2\bar{q} - p$

Thus $n_s(G) + n_s(\bar{G}) \leq 2(q + \bar{q}) - 2p$

$\leq p(p-1) - 2p$

$= p(p-3)$

Suppose the bound is attained, then $n_s(G) = 2q - p$and $n_s(\bar{G}) = 2\bar{q} - p$. This implies $q$and $\bar{q} < p$. Hence G and $\bar{G}$ are trees. i.e.$G = P_4$

Now we will establish a relation between Split Proximity number and maximum Proximity number.

Theorem 2.13 LetG be a graph with $\beta_\circ(G) \geq 3$ and possess no triangles.

Then, $n_s(G) \leq n_m(G)$.....................................(16)

Proof : Let S be a maximal Proximity set of G.Then $< V - S >$ is totally disconnected with at least two vertices. Thus S is a Split Proximity set. Hence (16) holds.

Theorem H [7] For any graph G,

$$n_m(G) \leq \alpha_\circ(G) + 1$$

Theorem 2.14 Let$G$ be a graph without triangle, then

$$n_m(G) \leq n_s(G) + 1 \quad \text{.............................(17)}$$

Proof : The Proof of (17)follows from (9) and Theorem 5.H [7].

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